Week 12

1 Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by

$$T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1}).$$

Find a formula for $T^*(z_1, \ldots, z_n)$.

Sol^n Recall Suppose $T \in \mathcal{L}(V,W)$. The <u>adjoint</u> of T is the map $T^*: W \to V$ defined by

$\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V \quad \forall v \in V \ w \in W$

To compute T^* it suffices to compute $T^* e_i$ for a basis $\{e_1, \cdots, e_n\}$. Take $\{e_1, \cdots, e_n\}$ the standard basis for $\mathbb{F}^n$

Then $\langle Te_j, e_i \rangle = \langle e_j, T^* e_i \rangle$

$$\langle e_i, Te_j \rangle = \begin{cases} \langle e_i, e_{j+1} \rangle = \delta_{i,j+1} & \text{if } j < n \\ \langle e_i, 0 \rangle = 0 & \text{if } j = n \end{cases}$$

$\therefore \ T^* e_i = \sum_{j=1}^{n} \langle T^* e_i, e_j \rangle e_j = \sum_{j=1}^{n} \delta_{i,j+1} e_j = e_{i-1}$

$\therefore \ T^*(w_1, \cdots, w_n) = (w_2, w_3, \cdots, w_n, 0)$.

Q2 Let $V = P_1(\mathbb{R})$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(t) g(t)\, dt$.

Let $T(f) = f' - f$ and $g(t) = 3t + 1$. Compute $T^*(g)$.

Sol^n Write $h := T^*(g) \in V$ and $h(t) = at + b$ for some $a, b \in \mathbb{R}$

For $f \in V$ $\quad \langle h, f \rangle = \langle T^*(g), f \rangle = \langle g, Tf \rangle$

Take $f = 1$, $2b = \int_{-1}^{1} (at+b)\, dt = \langle h, 1 \rangle = \langle g, T1 \rangle = \langle 3t+1, -1 \rangle$

$= \int_{-1}^{1} (-3t - 1)\, dt = -2$

$f = t$ $\quad \frac{2a}{3} = \int_{-1}^{1} (at+b) t\, dt = \langle h, t \rangle = \langle g, Tt \rangle = \langle 3t+1, 1-t \rangle$

$= \int_{-1}^{1} -3t^2 + 2t + 1\, dt = 0$

$\therefore \ b = -1 \quad a = 0 \quad$ and $\quad T^*(g) = -1$.

Q3. Let $V = P_2(\mathbb{R})$ Define the following inner products

$\langle f, g\rangle_1 = \int_{-1}^{1} f(x)g(x)dx$, $\langle f, g\rangle_2 = \int_{-1}^{1} f(x)g(x)(1-x^2)dx$, $f, g \in V$

Suppose $T \in \mathcal{L}(V)$ is a lin. op. def. by $T(f) = x\frac{df}{dx} + f(0)$

Consider the following operators

$\langle T_1 f, g\rangle_1 = \langle f, Tg\rangle_1$ $\langle T_2 f, g\rangle_2 = \langle f, Tg\rangle_2$ $f, g \in V$

Compute $T_i(ax^2+bx+c)$ for $i=1,2$ $a, b, c \in \mathbb{R}$

Sol^n $T(1) = 1$ $T(x) = x$ $T(x^2) = 2x^2$

$$\int_{-1}^{1} x^n dx = \begin{cases} \frac{2}{n+1} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \qquad \int_{-1}^{1} x^n(1-x^2)dx = \begin{cases} \frac{4}{(n+1)(n+3)} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

Note that $\langle tx^2+ux+v, px^2+qx+r\rangle_1$

$= \int_{-1}^{1} tpx^4 + (\cdots)x^3 + (tr+uq+vp)x^2 + (\cdots)x + vr\,dx$

$= \frac{2}{5}tp + \frac{2}{3}(tr+uq+vp) + 2vr$

If $T_1(ax^2+bx+c) = tx^2+ux+v$ for some $t, u, v \in \mathbb{R}$

Solve t, u, v in terms of a, b, c:

$\frac{2}{5}tp + \frac{2}{3}(tr+uq+vp) + 2vr = \langle T_1(ax^2+bx+c), px^2+qx+r\rangle_1$

$= \langle ax^2+bx+c, T(px^2+qx+r)\rangle_1$

$= \langle ax^2+bx+c, 2px^2+qx+r\rangle_1$

$= \frac{4}{5}ap + \frac{2}{3}(ar+bq+2cp) + 2cr$

Since p, q, r are arbitrary, we have

$\frac{2}{5}t + \frac{2}{3}v = \frac{4}{5}a + \frac{4}{3}c$ coeff. of p

$\frac{2}{3}u = \frac{2}{3}b$ coeff. of q

$\frac{2}{3}t + 2v = \frac{2}{3}a + 2c$ coeff. of r

$\therefore u=b \qquad 3t+5v=6a+10c$

$$t+3v = a+3c$$

$$\Rightarrow (9-5)v = (3-6)a+(9-10)c = -3a-c$$

$$\therefore v = \frac{-3}{4}a - \frac{1}{4}c$$

$$t = a+3c-3\left(\frac{-3}{4}a-\frac{1}{4}c\right)$$

$$= \frac{13}{4}a+\frac{15}{4}c$$

$$\therefore T_1(ax^2+bx+c) = \left(\frac{13}{4}a+\frac{15}{4}c\right)x^2 + bx + \left(\frac{-3}{4}a-\frac{1}{4}c\right)$$

Similarly $\langle tx^2+ux+v, px^2+qx+r\rangle_2$

$$= \frac{4}{35}tp + \frac{4}{15}(tp+uq+vp)+\frac{4}{3}vr$$

If $T_2(ax^2+bx+c) = tx^2+ux+v$

$$\frac{4}{35}tp + \frac{4}{15}(tr+uq+vp)+\frac{4}{3}vr$$

$$= \langle T_2(ax^2+bx+c), px^2+qx+r\rangle$$

$$= \langle ax^2+bx+c, 2px^2+qx+r\rangle$$

$$= \frac{4}{35}a(2p)+\frac{4}{15}(ar+bq+c(2p))+\frac{4}{3}cr$$

$$\therefore \frac{4}{35}t+\frac{4}{15}v = \frac{4}{35}a+\frac{8}{15}c$$

$$\frac{4}{15}u = \frac{4}{15}b$$

$$\frac{4}{15}t+\frac{4}{3}v = \frac{4}{15}a+\frac{4}{3}c$$

$$\therefore 3t+7v = 6a+14c$$

$$u=b$$

$$t+5v = a+5c$$

$$(15-7)v = (3-6)a+(15-14)c$$

$$v = \frac{-3}{8}a+\frac{1}{8}c$$

$$t = \left(\frac{15}{8}+1\right)a+\left(5-\frac{5}{8}\right)c = \frac{23}{8}a+\frac{35}{8}c$$

$$\therefore T_2(ax^2+bx+c) = \left(\frac{23}{8}a+\frac{35}{8}c\right)x^2 + x + \left(\frac{-3}{8}a+\frac{1}{8}c\right)$$

Q4. Let V be a fin. dim'l inner prod. sp. $T \in \mathcal{L}(V)$

Suppose $\alpha = (u_1, \cdots, u_n)$ is an O.N.B.
$\beta = (v_1, \cdots, v_n)$ is a basis (not necessarily orthonormal)
$x, y \in V$

Then $\exists\, a_i, b_i, x_i, y_i \in \mathbb{F}$ s.t.
$x = \sum_{i=1}^n a_i u_i = \sum_{i=1}^n x_i v_i$ and $y = \sum_{i=1}^n b_i u_i = \sum_{i=1}^n y_i v_i$

$a_i = \langle x, u_i \rangle \quad b_i = \langle y, u_i \rangle$.

$$\langle x, y \rangle = \langle \sum_i a_i u_i, \sum_j b_j u_j \rangle = \sum_i \sum_j \underline{a_i \bar{b}_j} \langle u_i, u_j \rangle$$
$$= \sum_i \sum_j a_i \bar{b}_j \delta_{ij} = \sum_i a_i \bar{b}_i$$

$$\therefore [\langle x, y \rangle] = [\bar{b}_1 \ \bar{b}_2 \ \cdots \ \bar{b}_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \mathcal{M}(y, \alpha)^* \mathcal{M}(x, \alpha)$$

Note that $\mathcal{M}(x, \alpha) = \mathcal{M}(I_V, \beta, \alpha)\mathcal{M}(x, \beta)$

$$\therefore [\langle x, y \rangle] = (\mathcal{M}(I_V, \beta, \alpha)\mathcal{M}(y, \beta))^* \mathcal{M}(I_V, \beta, \alpha)\mathcal{M}(x, \beta)$$
$$= \mathcal{M}(y, \beta)^* \mathcal{M}(I_V, \beta, \alpha)^* \mathcal{M}(I_V, \beta, \alpha)\mathcal{M}(x, \beta)$$

$$\mathcal{M}(I_V, \beta, \alpha) = \begin{bmatrix} \mathcal{M}(v_1, \alpha) & \cdots & \mathcal{M}(v_n, \alpha) \end{bmatrix} = \begin{bmatrix} \langle v_1, u_1 \rangle & \cdots & \langle v_n, u_1 \rangle \\ \vdots & & \\ \langle v_1, u_n \rangle & \cdots & \langle v_n, u_n \rangle \end{bmatrix}$$

$(\mathcal{M}(I_V, \beta, \alpha))_{ij} = \langle v_j, u_i \rangle$

$$\therefore H := \mathcal{M}(I_V, \beta, \alpha)^* \mathcal{M}(I_V, \beta, \alpha)$$
$$= \begin{bmatrix} \overline{\langle v_1, u_1 \rangle} & \cdots & \overline{\langle v_1, u_n \rangle} \\ & & \\ \overline{\langle v_n, u_1 \rangle} & & \overline{\langle v_n, u_n \rangle} \end{bmatrix} \begin{bmatrix} \langle v_1, u_1 \rangle & \cdots & \langle v_n, u_1 \rangle \\ \vdots & & \vdots \\ \langle v_1, u_n \rangle & \cdots & \langle v_n, u_n \rangle \end{bmatrix}$$

$$H_{ij} = \sum_{k=1}^n \overline{\langle v_i, u_k \rangle} \langle v_j, u_k \rangle = \langle v_j, v_i \rangle$$

Note $(H^*)_{ij} = \bar{H}_{ji} = \overline{\langle v_i, v_j \rangle} = \langle v_j, v_i \rangle = H_{ij} \quad \therefore H^* = H$ as matrices

Since H is a product of two invertible matrices, it is invertible

If $T \in \mathcal{L}(V)$, then $\langle Tx, y\rangle = \langle x, T^* y\rangle$

$$[\langle Tx, y\rangle] = \mathcal{M}(y,\beta)^* \cdot H \cdot \mathcal{M}(Tx,\beta)$$
$$= \mathcal{M}(y,\beta)^* \cdot H\, \mathcal{M}(T,\beta)\, \mathcal{M}(x,\beta)$$
$$[\langle x, T^* y\rangle] = \mathcal{M}(T^* y,\beta)\ H\, \mathcal{M}(x,\beta)$$
$$= (\mathcal{M}(T^*,\beta)\mathcal{M}(y,\beta))^* H\, \mathcal{M}(x,\beta)$$
$$= \mathcal{M}(y,\beta)^*\, \mathcal{M}(T^*,\beta)^*\, H\, \mathcal{M}(x,\beta)$$

Since x, y arbitrary, take $x = v_j$ $y = v_i$

$$e_i^*\, \mathcal{M}(T^*,\beta)^* H\, e_j = e_i^*\, H\, \mathcal{M}(T,\beta)\, e_j$$
$$\therefore\ \mathcal{M}(T^*,\beta)^* H = H\,\mathcal{M}(T,\beta)$$
$$\therefore\ \mathcal{M}(T^*,\beta)^* = H\,\mathcal{M}(T,\beta) H^{-1}$$
$$\mathcal{M}(T^*,\beta) = (H\,\mathcal{M}(T,\beta)H^{-1})^* = (H^*)^{-1}\mathcal{M}(T,\beta)^* H^*$$
$$= H^{-1}\mathcal{M}(T,\beta)^* H$$